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On quotient structure of Takasaki quandles II

Yongju Bae* and Seongjeong Kim†

Department of Mathematics,
College of Natural Sciences,
Kyungpook National University,
Daegu 702-701, Korea
**ybae@knu.ac.kr*
†ksj19891120@gmail.com

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ABSTRACT

A Takasaki quandle $(T(G), *)$ is a quandle under the binary operation $*$ defined by $a*b = 2b-a$ for an abelian group $(G, +)$. In this paper, we will show that if a subquandle X of a Takasaki quandle G is a image of subgroup of G under a quandle automorphism of $T(G)$, then the set $\{X * g \,|\, g \in G\}$ is a quandle under the binary operation $*'$ defined by $(X * g) *' (X * h) = X * (g * h)$. On the other hand, the quotient structure studied in [On quotients of quandles, *J. Knot Theory Ramifications* **19**(9) (2010) 1145–1156] can be applied to the Takasaki quandles. In this paper, we will review the quotient structure studied in [On quotients of quandles, *J. Knot Theory Ramifications* **19**(9) (2010) 1145–1156], and show that the quotient quandle coincides with the quotient quandle defined by Bunch, Lofgren, Rapp and Yetter in [On quotients of quandles, *J. Knot Theory Ramifications* **19**(9) (2010) 1145–1156] for connected Takasaki quandles.

Keywords: Quandle; quotient quandle; Takasaki quandle; subquandle.

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1. Introduction

In 1942, Takasaki [11] introduced the notion of *kei*. In 1980s, Joyce [4] and Matveev [5] introduced the definition of *quandles* independently. A quandle is an algebraic object with a binary operation satisfying three axioms derived from Reidemeister moves. In Joyce's terminology, a kei is an *involutory quandle*. A Takasaki quandle is one of examples of keis defined by abelian groups. In [4, 5], it was shown that classical knots can be almost completely classified by the fundamental quandles of knots. But the direct computation of the fundamental quandles of knots is very complicated. Many mathematicians study algebraic structures for quandles. Ferman, Nowik and Teicher [3] and Nelson [6] studied algebraic structures of Alexander quandles, while Nelson and Wong [7] and Roszkowska-Lech [9] studied about a decomposition of quandles, derived from the inner quandle automorphisms.

In [10], Roszkowska-Lech classified subdirectly irreducible abelian quandles by using the decomposition. In [2], Bunch, Lofgren, Rapp and Yetter defined a quotient structure of quandles in terms of the inner quandle automorphism groups of quandles and studied its properties. In [8], Roszkowska-Lech proved that for an abelian group G of odd order, there is a congruence relation on $T(G)$ if and only if there is a congruence relation on G. In [1], the authors studied a quotient structure of Takasaki quandles in terms of subquandles.

In this paper, we will show that if X is a subquandle of a Takasaki quandle $T(G)$ which is the image of a subgroup of G under any quandle automorphism of $T(G)$, then the set $\{X * g \,|\, g \in G\}$ is a quandle, called the quotient quandle, under the binary operation

$$(X * g) *' (X * h) = X * (g * h).$$

Furthermore we will show that the above quandle coincides with the quotient quandle defined by Bunch, Lofgren, Rapp and Yetter for connected Takasaki quandles.

2. Basic Definitions and Properties

A *quandle* is a set Q equipped with a binary operation $* : Q \times Q \to Q$ satisfying the following three axioms:

(1) For all $x \in Q$, $x * x = x$.
(2) For all $x, y \in Q$, $\exists!\ z \in Q$ such that $z * x = y$ (denote $z = y \,\bar{*}\, x$).
(3) For all $x, y, z \in Q$, $(x * y) * z = (x * z) * (y * z)$.

A set X with a binary operation $*$ defined by $a * b = a$ is a quandle, which is called the *trivial quandle*. An abelian group $(G, +)$ with a binary operation $*$ defined by $a * b = 2b - a$ is a quandle, which is called the *Takasaki quandle* of an abelian group $(G, +)$, which would be denoted by $(T(G), *)$. For a module M over the Laurent polynomial ring $\mathbb{Z}[t^{\pm 1}]$, $(M, *)$ with the binary operation $*$ defined by $a * b = ta + (1 - t)b$ is a quandle, which is called an *Alexander quandle*.

By the second axiom of a quandle, $\bar{*} : Q \times Q \to Q$ is also a binary operation. The second axiom of a quandle is equivalent with that two binary operations $*$ and $\bar{*}$ satisfy the following property; for all $x, y \in Q$,

$$(x * y) \,\bar{*}\, y = x = (x \,\bar{*}\, y) * y.$$

A function f from a quandle $(Q, *)$ to an another quandle $(R, *')$ is a *quandle homomorphism* if $f(x * y) = f(x) *' f(y)$. A quandle homomorphism is called a *quandle isomorphism* if it is bijective. A quandle isomorphism from Q to itself is called a *quandle automorphism*. By the second axiom and the third axiom of a quandle, it is clear that a mapping $*x : Q \to Q$ defined by $*x(y) = y * x$ is a quandle automorphism. Indeed,

$$(*x)^{-1} = \bar{*}x.$$

Let $\mathrm{Inn}(Q)$ be the subgroup of the group of all quandle automorphisms which is generated by $\{- * q, - \bar{*}\, q \mid q \in Q\}$. We call $\mathrm{Inn}(Q)$ the *inner quandle automorphism group* of Q. It is clear that if two abelian groups $(G,+)$ and $(H,+)$ are isomorphic as a group, then two Takasaki quandles $(T(G),*)$ and $(T(H),*)$ are isomorphic as a quandle, since the quandle operation $*$ of a Takasaki quandle is derived from the group operation. Generally, a quandle isomorphism is not a group isomorphism. For example, for $a \in G$, the function $f : T(G) \to T(G)$ defined by $f(x) = x+a$ is a quandle automorphism of $T(G)$ but it is not a group automorphism of G.

A quandle $(Q,*)$ is called a *connected quandle* if the action of $\mathrm{Inn}(Q)$ on X is transitive, in other words, for any $q, r \in Q$, there exists $\sigma \in \mathrm{Inn}(Q)$ such that $q\sigma = \sigma(q) = r$. Notice that for $\sigma \in \mathrm{Inn}(Q)$, there exist $r_1, r_2, \ldots, r_n$ in Q such that

$$q\sigma = \sigma(q) = (\cdots((q *_1 r_1) *_2 r_2)\cdots) *_n r_n$$

for all $q \in Q$, $*_i \in \{*, \bar{*}\}$ for all $i \in \{1, 2, \ldots, n\}$. From this observation, one can see that if a quandle is obtained from an already known algebraic structure, then $q\sigma$ can be represented by the operation of the algebraic structure, since the quandle operation is applied finitely many times. The connectedness of a Takasaki quandle depends, especially, on the characteristic of the underlying abelian group.

Lemma 2.1. *A Takasaki quandle $(T(G),*)$ is connected if and only if $2G = G$.*

Proof. Since $(x * y) * y = 2y - (2y - x) = x = (x * y)\,\bar{*}\, y$ for all $x, y \in G$, $\bar{*}x = *x$ for every $x \in G$. Hence $(T(G),*)$ is connected if and only if for all $q, r \in G$, there exist $r_1, r_2, \ldots, r_n$ in G such that $r = (\cdots((q * r_1) * r_2)\cdots) * r_n$. Notice that

$$(\cdots((q * r_1) * r_2)\cdots) * r_n = (-1)^n q + 2\left(\sum_{i=1}^{n}(-1)^{n-i} r_i\right)$$

and $\sum_{i=1}^{n}(-1)^{n-i} r_i \in G$.

Suppose $(T(G),*)$ is connected. Then for all $r \in G$, there exist $r_1, r_2, \ldots, r_n$ in G such that $r = (-1)^n 0 + 2(\sum_{i=1}^{n}(-1)^{n-i} r_i) = 2(\sum_{i=1}^{n}(-1)^{n-i} r_i) \in 2G$. Hence $G \subset 2G$. Since $2G \subset G$, $2G = G$. Conversely, if $2G = G$, then for $q, r \in G$, there exists $s \in G$ such that $q + r = 2s$, that is, $r = 2s - q = q * s$. Therefore $(T(G),*)$ is connected. □

A quandle is said to be *abelian* if $(x*y)*(z*w) = (x*z)*(y*w)$ for $x, y, z, w \in T(G)$. By the definition of an Alexander quandle, we can see the following lemma.

Lemma 2.2. *Every Alexander quandle is abelian. In particular, a Takasaki quandle is abelian.*

Proof. For elements $x, y, z, w \in M$,

$$\begin{aligned}(x * y) * (z * w) &= t(tx - (1-t)y) + (1-t)(tz + (1-t)w) \\ &= t^2 x - t(1-t)y + t(1-t)z + (1-t)^2 \\ &= t(tx - (1-t)z) + (1-t)(ty + (1-t)w) \\ &= (x * z) * (y * w).\end{aligned}$$

Therefore, $(M, *)$ is an abelian quandle. In the case of $t = -1$, $(M, *)$ is a Takasaki quandle and hence Takasaki quandles are also abelian. $\square$

A subset X of $(Q, *)$ is called a *subquandle* of Q if X itself is a quandle under $*$. Indeed, every subset $X \subset Q$ which is closed under $*$ and $\bar{*}$ is a subquandle because every subset of Q satisfies the first and the third axioms of a quandle. Thus we have Proposition 2.3.

Proposition 2.3. *Let $(Q, *)$ be a quandle. A subset X of Q is a subquandle of Q if and only if X is closed under $*$ and $\bar{*}$.*

Corollary 2.4. *Let $(T(G), *)$ be the Takasaki quandle of an abelian group $(G, +)$. A subset X is a subquandle of $T(G)$ if and only if it is closed under $*$.*

Proof. As mentioned in the proof of Lemma 2.1, $\bar{*}x = *x$ for every $x \in X$ and hence the statement is followed from Proposition 2.2. $\square$

Let $f : T(G) \to T(G)$ is a quandle automorphism. If $X \subset T(G)$ is a subquandle of $T(G)$, then one can easily see that $f(X)$ is also a subquandle of $T(G)$.

Corollary 2.5. *Let $(T(G), *)$ be the Takasaki quandle of an abelian group $(G, +)$. If X is a subgroup of G, then X is a subquandle of $(T(G), *)$.*

Proof. Let X be a subgroup of G. Let $x, y \in X$. Since X is a subgroup of G, $x * y = 2y - x \in X$. By Corollary 2.4, X is a subquandle of $(T(G), *)$. $\square$

Corollary 2.6. *Let $(T(G), *)$ be the Takasaki quandle of an abelian group $(G, +)$. If X is a subgroup of G, then $X + g$ is a subquandle of $(T(G), *)$ for all $g \in G$.*

Proof. Since $f : T(G) \to T(G)$ defined by $f(x) = x + g$ is quandle automorphism for $g \in G$, $X + g$ is also a subquandle. $\square$

3. Quotient Quandle Defined from Subquandles

In [1], the authors studied about necessary and sufficient condition for subquandles containing 0 to define a quotient quandle of connected Takasaki quandles. In this section we will study for subquandles which may not contain 0. Firstly we review the following two lemmas.

Lemma 3.1 ([1]). *Let $(T(G), *)$ be the Takasaki quandle of an abelian group $(G, +)$. Then every subquandle X containing the identity 0 is an ideal of G as a $\mathbb{Z}$-module. Indeed, $kx \in X$ for $x \in X$ and $k \in \mathbb{Z}$.*

Lemma 3.2 ([1]). *Let $(T(G), *)$ be the Takasaki quandle of an abelian group $(G, +)$. Let X be a subquandle of $(T(G), *)$ containing 0. If X is a subgroup of $(G, +)$, then, for $a, b \in G$,*

$$(X * a) \cap (X * b) = \emptyset \quad \text{or} \quad X * a = X * b.$$

Moreover, if $(T(G), *)$ is a connected quandle, then the converse is also true.

Lemma 3.2 can be improved for subquandles not containing 0 as follows.

Lemma 3.3. *Let $(T(G), *)$ be the Takasaki quandle of an abelian group $(G, +)$ and Y a subquandle of $T(G)$. If there is a quandle automorphism f of $T(G)$ and a subgroup X of G such that $Y = f(X)$, then, for $a, b \in G$, either*

$$(Y * a) \cap (Y * b) = \emptyset \quad \text{or} \quad Y * a = Y * b.$$

Moreover, if $(T(G), *)$ is a connected quandle, then the converse is also true.

Proof. Assume that $(Y * a) \cap (Y * b) \neq \emptyset$. Since f is a quandle automorphism of $T(G)$,

$$\begin{aligned} X * f^{-1}(a) \cap X * f^{-1}(b) &= f^{-1}(Y) * f^{-1}(a) \cap f^{-1}(Y) * f^{-1}(b) \\ &= f^{-1}((Y * a) \cap (Y * b)) \\ &\neq \emptyset. \end{aligned}$$

Since X is a subgroup of G, by Lemma 3.2, $X * f^{-1}(a) = X * f^{-1}(b)$. Therefore $Y * a = f(X * f^{-1}(a)) = f(X * f^{-1}(b)) = Y * b$.

Conversely, assume that $(Y * a) \cap (Y * b) = \emptyset$ or $Y * a = Y * b$ for $a, b \in G$. We will construct a subgroup X of G and a quandle homomorphism f of $T(G)$ such that $Y = f(X)$. Let $g \in Y$ be fixed. By Corollary 2.6, $X = Y - g$ is also a subquandle containing 0. Let $f : T(G) \to T(G)$ be a quandle automorphism defined by $f(x) = x + g$. Since f is a quandle automorphism and $X = f^{-1}(Y)$, the assumption implies that $(X * a) \cap (X * b) = \emptyset$ or $X * a = X * b$ for $a, b \in G$. Since X contains 0 and $T(G)$ is connected, by Lemma 3.2, X is a subgroup of G and hence $Y = f(X)$ for some subgroup X of G. $\square$

Remark 3.4. In the previous lemma, the set $\{X * a\}_{a \in G}$ is not a partition of G, in general. For example, consider $G = \mathbb{Z}_8$. Then $X = \{0, 2, 4, 6\}$ is a subgroup of G, but $X * x = X$ for all $x \in \mathbb{Z}_8$. That is, $\bigcup_{a \in G}(X * a) = X \neq G$.

The following lemma says that $\{X * a\}_{a \in G}$ gives a partition for a connected Takasaki quandle $T(G)$.

Lemma 3.5. *Let $(T(G), *)$ be the Takasaki quandle of an abelian group $(G, +)$ and X a subquandle of $T(G)$. If $(T(G), *)$ is a connected quandle, then $G = \bigcup_{a \in G}(X * a)$.*

Proof. Let X be a subquandle of a connected Takasaki quandle $(T(G), *)$. Fix an element $g \in X$. It is trivial that $\bigcup_{a\in G}(X * a) \subset G$. Let x be an element in G. Since $(T(G), *)$ is connected, by Lemma 2.1, $2G = G$. Since $x + g \in G$, there exists h in G such that $2h = x + g$, i.e. $g * h = 2h - g = x$. Since $g \in X$, $x = g * h \in X * h$, i.e. $g \in G = \bigcup_{a\in G}(X * a)$. □

Corollary 3.6. *Let $(T(G), *)$ be the Takasaki quandle of an abelian group $(G, +)$ and Y a subquandle of $T(G)$. Assume that $T(G)$ is a connected quandle. Then $\{Y * a\}_{a\in G}$ gives a partition of G if and only if $Y = f(X)$ for some subgroup X of G and a quandle automorphism f of $T(G)$.*

Even though $\{Y * a\}_{a\in G}$ is not a partition for G in general, $\{Y * a\}_{a\in G}$ is the set of mutually disjoint subsets as seen in Lemma 3.3. The following theorem says that one can give a natural quandle structure on $\{Y * a\}_{a\in G}$.

Theorem 3.7. *Let $(T(G), *)$ be the Takasaki quandle of an abelian group $(G, +)$ and X a subgroup of G. Let $f : T(G) \to T(G)$ be a quandle automorphism and $Y = f(X)$. Define a binary operation $*'$ on $\{Y * a\}_{a\in G}$ by*

$$(Y * a) *' (Y * b) = Y * (a * b).$$

Then $(\{Y * a\}_{a\in G}, *')$ is a quandle. In fact, $(\{Y * a\}_{a\in G}, *')$ is isomorphic to $T(2(G/X))$ as a quandle. Denote $\{Y * a\}_{a\in G}$ by $T(G)/Y$.

Proof. Let X be a subgroup of G and $Y = f(X)$ for a quandle automorphism f of $T(G)$. Firstly we will show that the operation $*'$ is well-defined. Let $Y * a_1$, $Y * a_2$, $Y * b_1$ and $Y * b_2$ be elements of $\{Y * a\}_{a\in G}$. Assume that $Y * a_1 = Y * a_2$ and $Y * b_1 = Y * b_2$. It suffices to show that $Y * (a_1 * b_1) = Y * (a_2 * b_2)$. Since $T(G)$ is an abelian quandle and $Y = Y * Y$, we can see that

$$Y * (a_1 * b_1) = (Y * a_1) * (Y * b_1) \quad \text{and} \quad Y * (a_2 * b_2) = (Y * a_2) * (Y * b_2).$$

By the assumption, $(Y * a_1) * (Y * b_1) = (Y * a_2) * (Y * b_2)$ and hence $Y * (a_1 * b_1) = Y * (a_2 * b_2)$.

Now we will show that $*'$ satisfies the three axioms of a quandle. It is trivial that $*'$ satisfies the first axiom and the third axiom of a quandle. For the second axiom of a quandle, let $Y * a$ and $Y * b$ be arbitrary in $\{Y * a\}_{a\in G}$. Since $T(G)$ is a quandle, there exists $c \in G$ such that $c * a = b$. It is clear that $(Y * c) *' (Y * a) = Y * b$. For the uniqueness of $Y * c$, suppose that $(Y * d) *' (Y * a) = Y * b$. Since $Y * (c * a) = Y * (d * a)$, $X * (f^{-1}(c) * f^{-1}(a)) = X * (f^{-1}(d) * f^{-1}(a))$. There exists $x \in X$ such that $0 * (f^{-1}(c) * f^{-1}(a)) = x * (f^{-1}(d) * f^{-1}(a))$, or equivalently,

$$\begin{aligned} 4f^{-1}(a) - 2f^{-1}(c) &= 2(2f^{-1}(a) - f^{-1}(c)) \\ &= 2(2f^{-1}(a) - f^{-1}(d)) - x \\ &= 4f^{-1}(a) - 2f^{-1}(d) - x \end{aligned}$$

so that

$$0 * f^{-1}(d) = 2f^{-1}(d) = 2f^{-1}(c) - x = x * f^{-1}(c).$$

Since $f(0) * d = f(x) * c \in Y * c$, by Lemma 3.3, $(Y * c) = (Y * d)$. Therefore $(\{Y * a\}_{a \in G}, *')$ is a quandle.

Notice that $Y * a = 2a - Y = 2a - f(X) = f(2f^{-1}(a) - X)$. Define a function $\phi : T(G)/Y \to T(2(G/X))$ by $\phi(Y * a) = \phi(f(2f^{-1}(a) - X)) = 2f^{-1}(a) + X$. It is easy to show that ϕ is a well-defined bijective function. Moreover,

$$\begin{aligned}
\phi((Y * a) *' (Y * b)) &= \phi(Y * (a * b)) \\
&= 2f^{-1}(a * b) + X \\
&= 2(2f^{-1}(b)) - 2f^{-1}(a) + X \\
&= (2f^{-1}(a) + X) * (2f^{-1}(b) + X) \\
&= \phi(Y * a) * \phi(Y * b).
\end{aligned}$$

Hence $T(G)/Y$ is isomorphic to $T(2(G/X))$. □

Corollary 3.8. *Let $(T(G), *)$ be the Takasaki quandle of an abelian group $(G, +)$ and X a subgroup of G. Let $f : T(G) \to T(G)$ be a quandle automorphism and $Y = f(X)$. If $T(G)$ is connected, then $(T(G)/Y, *')$ is isomorphic to $T(G/X)$ as a quandle.*

Proof. Assume that $T(G)$ is connected. By Lemma 2.1, $2G = G$. That is, $G/X = 2G/X$. Therefore, by Theorem 3.7, $T(G)/Y$ is isomorphic to $T(G/X)$. □

Example 3.9. Consider the Takasaki quandle $(T(\mathbb{R}), *)$ of the abelian group $(\mathbb{R}, +)$. Since $\mathbb{Z}$ is a subgroup $\mathbb{R}$, $(T(\mathbb{R}/\mathbb{Z}), *')$ is a quandle. Since $2\mathbb{R} = \mathbb{R}$ and $(\mathbb{R}, +)$ is a connected quandle, by Corollary 3.8, $T(\mathbb{R})/\mathbb{Z}$ is isomorphic to $T(\mathbb{R}/\mathbb{Z}) \cong T(S^1)$.

Remark 3.10. For an Alexander quandle $(M, *)$, we can show that if $S \subset M$ is a submodule and $f : M \to M$ is a quandle automorphism, then $\{f(S) * a\}_{a \in M}$ is a set of disjoint subsets of M. Moreover, $\{f(S) * a\}_{a \in M}$ is a quandle under the binary operation $*'$ defined by $(f(S) * a) *' (f(S) * b) = f(S) * (a * b)$. But we do not know that "submodule" is the necessary and sufficient condition to define a quotient quandle of Alexander quandles.

4. Relationship Between the Quotient Quandle from a Subquandle and the BLRY Quotient Quandle

In [2], Bunch, Lofgren, Rapp and Yetter introduced a quotient structure for quandles defined by the group of inner quandle automorphisms. It can be applied for every quandle to get a quotient quandle. In Sec. 3, we studied quotient quandle of

Takasaki quandles defined by subquandles. In this section we will review the construction of the quotient quandle in [2] and will compare two quotient structures of Takasaki quandles.

Consider the right action on a quandle $(Q, *, \bar{*})$ by $\mathrm{Inn}(Q)$ defined by $q\sigma = \sigma(q)$ for $\sigma \in \mathrm{Inn}(Q)$ and for $q \in Q$. If Q is connected, then there is the only one orbit. If N is a subgroup of $\mathrm{Inn}(Q)$, then there is another right action on Q by N defined by $q\sigma = \sigma(q)$ for $\sigma \in N$ and for $q \in Q$. Define an equivalence relation $\sim_N$ by $q \sim_N r$ if and only if q and r are on the same orbit of the right action by N, that is,

$$q \sim_N r \Leftrightarrow q\sigma = r \quad \text{for some } \sigma \in N.$$

It is easy to show that this relation is a well-defined equivalence relation. Furthermore, we have the following proposition.

Proposition 4.1 ([2]). *Let $(Q, *, \bar{*})$ be a quandle. Let N be a subgroup of $\mathrm{Inn}(Q)$. Then $\sim_N$ is a congruence relation if and only if N is normal in $\mathrm{Inn}(Q)$.*

Since $\sim_N$ is a congruence relation, $\{qN\}_{q\in Q}$ is a quandle with the binary operation $*'$ defined by

$$(qN) *' (rN) = (q * r)N.$$

We denote $\{qN\}_{q\in Q}$ by Q/N and call it *BLRY quotient quandle* of Q by N.

Proposition 4.2 ([2]). *Let $(Q, *_Q, \bar{*}_Q)$ and $(R, *_R, \bar{*}_R)$ be two quandles. Let $h : Q \to R$ be a surjective quandle homomorphism. Let $N = \ker(\mathrm{Inn}_Q(h))$ and g_N the canonical quandle homomorphism from Q to $Q/\!\sim_N$. Then there exists a quandle homomorphism $f : Q/\!\sim_N \to R$ such that $h = f \circ g_N$ and $\mathrm{Inn}_Q(f)$ is a group isomorphism.*

To study the quotient structure defined by inner quandle automorphisms, we need to know the properties of inner quandle automorphisms and the properties of normal subgroups of $\mathrm{Inn}(Q)$.

Lemma 4.3. *Let $(Q, *, \bar{*})$ be a quandle. For $\sigma \in \mathrm{Inn}(Q)$ and $q \in Q$,*

$$\sigma \circ (*q) \circ \sigma^{-1} = *q\sigma.$$

Proof. For $\sigma \in \mathrm{Inn}(Q)$ and $q \in Q$, $\sigma \circ (*q) \circ \sigma^{-1}(r) = \sigma \circ (*q) \circ (\sigma^{-1}(r)) = \sigma(\sigma^{-1}(r) * q) = r * \sigma(q) = r * q\sigma = *q\sigma(r)$ for all $r \in Q$. Hence, $\sigma \circ (*q) \circ \sigma^{-1} = *q\sigma$. $\square$

Lemma 4.4. *Let $(Q, *, \bar{*})$ be a quandle. Let N be a normal subgroup of $\mathrm{Inn}(Q)$. Then qN is a subquandle of Q for $q \in Q$.*

Proof. To show that qN is a subquandle of Q for $q \in Q$, it suffices to show that $q\sigma * q\tau \in qN$ and $q\sigma \,\bar{*}\, q\tau \in qN$ for σ and τ in N. For σ and τ in N, $q\sigma * q\tau = ((((q \,\bar{*}\, q)\sigma)\tau) * q)\tau^{-1}$ by Lemma 4.3. Since N is a normal subgroup in

$\mathrm{Inn}(Q)$,$((((-\bar{*}\,q)\sigma)\tau) * q) \in N$ and hence $((((-\bar{*}\,q)\sigma)\tau) * q)\tau^{-1} \in N$. Therefore $q\sigma * q\tau \in qN$. Similarly, $q\sigma \,\bar{*}\, q\tau \in qN$. □

Lemma 4.5. *Let* $(Q, *, \bar{*})$ *be a connected quandle. Let* N *be a proper normal subgroup of* $\mathrm{Inn}(Q)$. *Then neither* $*q$ *nor* $\bar{*}q$ *is in* N *for all* $q \in Q$.

Proof. Suppose that $*q \in N$ for some $q \in Q$. Since N is a normal subgroup of $\mathrm{Inn}(Q)$, $\sigma^{-1} \circ (*q) \circ \sigma \in N$, for any $\sigma \in \mathrm{Inn}(Q)$. Since Q is connected and $\sigma^{-1} \circ (*q) \circ \sigma = *q\sigma$, for any $r \in Q$, $r = \sigma(q)$ for some $\sigma \in \mathrm{Inn}(Q)$. Therefore $*r \in N$ for any $r \in Q$. Since N is a subgroup, $\bar{*}q \in N$ for all $q \in Q$. Hence $N = \mathrm{Inn}(Q)$. This is a contradiction. By similar argument, one can see $\bar{*}q \notin N$. □

In Sec. 3, we studied our quotient quandle $T(G)/Y$ defined with subquandles and showed that necessary and sufficient condition for subquandles to define a partition in a connected quandle. From now on, we will compare the BLRY quotient quandle $T(G)/N$ and our quotient quandle $T(G)/X$ for a connected Takasaki quandle.

To compare those two quotient structures $T(G)/X$ and $T(G)/N$ of Takasaki quandles, we need to figure out the form of $\mathrm{Inn}_Q(T(G))$. Since Takasaki quandles are derived from the group structure, the following lemma can be shown.

Lemma 4.6. *Let* $(T(G), *)$ *be a Takasaki quandle. Then* $\mathrm{Inn}_Q(T(G)) = \{\psi_{2g}, \phi_{2g} \,|\, g \in G\}$, *where* $\psi_g(x) = -x + g$ *and* $\phi_g(x) = x + g$.

Proof. As mentioned in Lemma 2.1, for $\sigma \in \mathrm{Inn}_Q(T(G))$, $\sigma(x) = (\cdots((x * r_1) * r_2)\cdots) * r_n = (-1)^n x + 2(\sum_{i=1}^{n}(-1)^{n-i} r_i)$ for some $r_1, r_2, \ldots, r_n$ in Q. Put $g = \sum_{i=1}^{n}(-1)^{n-i} r_i$. Then

$$\sigma(x) = \begin{cases} x + 2g, & \text{if } n \text{ is even,} \\ -x + 2g, & \text{if } n \text{ is odd.} \end{cases}$$

Therefore every $\sigma \in \mathrm{Inn}_Q(T(G))$ has the form of ψ_{2g} or ϕ_{2g} for some $g \in G$. □

For a connected quandle Q, if $N = \mathrm{Inn}(Q)$, then Q/N has the only one element. Therefore we will consider a proper normal subgroup N of $\mathrm{Inn}_Q(T(G))$.

Corollary 4.7. *Let* $(T(G), *)$ *be a connected Takasaki quandle. Let* N *be a proper normal subgroup of* $\mathrm{Inn}_Q(T(G))$. *Then* $N = \{\phi_{2g} \,|\, g \in H\}$ *for some subgroup* H *of* G. *Moreover*, $0N$ *is a subgroup of* G.

Proof. Since $\psi_{2g}(x) = -x + 2g = x * g$, by Lemma 4.5, $\psi_{2g} \notin N$ for every $g \in G$. Let $H = \{g \in G \,|\, \phi_{2g} \in N\}$. Then $N = \{\phi_{2g} \,|\, g \in H\}$. We will show that H is a subgroup of G. Let $g, h \in H$. By the definition of H, ϕ_{2g} and ϕ_{2h} are in N. Since N is a subgroup of $\mathrm{Inn}(Q)$, $\phi_{2g}\circ\phi_{2h}^{-1} \in N$. Since $\phi_{2g}\circ\phi_{2h}^{-1}(x) = x+2(g-h) = \phi_{2(g-h)}(x)$, by the definition of H, $g-h \in H$. Hence H is a subgroup. Clearly, $0N = 2H$ and it is a subgroup of G. ☐

Lemma 4.8. *Let* $(T(G), *)$ *be a connected Takasaki quandle and* $H \subset G$ *a subgroup. Let* $N_H = \{\phi_g \mid g \in H\}$. *Then* N_H *is a proper normal subgroup of* $\mathrm{Inn}_Q(T(G))$. *Moreover*, $0N_H = H$.

Proof. First, we will show that N_H is a subgroup of $\mathrm{Inn}_Q(T(G))$. Let $\phi_g, \phi_h \in N_H$. Note that $\phi_h^{-1} = \phi_{-h}$. Then $\phi_g \circ \phi_h^{-1}(x) = \phi_g \circ \phi_{-h}(x) = x + (g-h) = \phi_{g-h}(x)$. Since H is a subgroup of G, $g-h \in H$. Hence $\phi_{g-h} \in N_H$. Therefore, N_H is a subgroup of $\mathrm{Inn}_Q(T(G))$.

Now we will show that $\sigma^{-1}\circ\phi_g\circ\sigma \in N_H$ for every $\sigma \in \mathrm{Inn}_Q(T(G))$. By Lemma 4.6, σ is either ψ_{2a} or ϕ_{2a} for some $a \in T(G)$. If $\sigma = \phi_{2a}$ for $a \in G$, then

$$\begin{aligned}\phi_{2a}^{-1}\circ\phi_g\circ\phi_{2a}(x) &= \phi_{2a}^{-1}\circ\phi_g(x+2a)\\ &= \phi_{2a}^{-1}(x+g+2a)\\ &= x+g+2a-2a\\ &= x+g\\ &= \phi_g(x).\end{aligned}$$

If $\sigma = \psi_{2a}$ for $a \in G$, then

$$\begin{aligned}\psi_{2a}^{-1}\circ\phi_g\circ\psi_{2a}(x) &= \psi_{2a}^{-1}\circ\phi_g(-x+2a)\\ &= \psi_a^{-1}(-x+g+2a)\\ &= -(-x+g+2a)+2a\\ &= x-g\\ &= \phi_{-g}(x).\end{aligned}$$

Therefore for any $\sigma \in \mathrm{Inn}(Q)$, $\sigma^{-1}\circ\phi_g\circ\sigma \in N_H$. Hence N_H is a normal subgroup of $\mathrm{Inn}_Q(T(G))$. ☐

By Corollary 4.7 and Lemma 4.8, for a subgroup $H \subset G$, there exists a proper normal subgroup N_H such that $0N_H = 2H$. Conversely, for a proper normal subgroup N of $\mathrm{Inn}_Q(T(G))$, there exists a subgroup H of G such that $N_H = N$.

Theorem 4.9. *Let* $(T(G), *)$ *be a connected Takasaki quandle. Let* H *be a subgroup of* G. *Then* $T(G)/H = T(G)/N_H$ *as a quandle.*

Proof. Let $(T(G), *)$ be a connected Takasaki quandle. Let H be a subgroup of G and $N_H = \{\phi_h \mid h \in H\}$. By Lemma 4.8, N_H is a normal subgroup of $\mathrm{Inn}_Q(T(G))$.

First we will show that $T(G)/H = T(G)/N_H$ as a set, i.e. $\{H * g\}_{g\in G} = \{kN_H\}_{k\in G}$. For $H*g \in \{H*g\}_{g\in G}$, $H*g = 2g-H = 2g+H = 2gN_H \in \{kN_H\}_{k\in G}$.

For $kN_H \in \{kN_H\}_{k\in G}$, $kN_H = \{k + h \,|\, h \in H\} = k + H$. Since $(T(G), *)$ is connected, $G = 2G$. There exists $g \in G$ such that $k = 2g$. Therefore $kN_H = k + H = 2g + H = H * g \in \{H * g\}_{g\in G}$.

Now we will show that they have exactly the same quandle operation. Let $H * g_1$ and $H * g_2$ be elements of $T(G)/H$ and let k_1N_H and k_2N_H elements of $T(G)/N_H$. Assume that $H * g_1 = k_1N_H$ and $H * g_2 = k_2N_H$. We can get $2g_1 + H = k_1 + H$ and $2g_2 + H = k_2 + H$.Then

$$\begin{aligned} k_1N_H * k_2N_H &= (k_1 * k_2)N_H \\ &= k_1 * k_2 + H \\ &= 2(g_1 * g_2) + H \\ &= H * (g_1 * g_2) \\ &= (H * g_1) *' (H * g_2). \end{aligned}$$

Hence $T(G)/H = T(G)/N_H$ as a quandle. □

Corollary 4.10. *Let $(T(G), *)$ be a connected Takasaki quandle. Let X be a subquandle of $T(G)$. Assume that $X = H + a$ for some subgroup H of G. Then $T(G)/X \cong T(G)/N_H$ as a quandle.*

Proof. Since $(T(G), *)$ is a connected Takasaki quandle, by Corollary 3.8, $T(G)/X \cong T(G/H) \cong T(G)/H$. By the above theorem, $T(G)/X \cong T(G)/H \cong T(G)/N_H$. □

Remark 4.11. BLRY quotient quandle can be defined for all quandles, but it is difficult to figure out the inner quandle automorphism group of quandles and its normal subgroups. Our quotient structure is defined by the familiar method that is used to define the quotient structure for other algebraic structures, e.g. a quotient group, a quotient ring and a quotient module, so that it is easy to understand and to calculate the quotient quandle. We defined our quotient structure for only Takasaki quandles, and are trying to apply this construction for other quandles.

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